

The natural metric in the Horrocks-Mumford bundle is not Hermitian-Einstein.

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Abstract. The Horrocks-Mumford bundle E is a famous stable complex vector bundle of rank 2 on 4-dimensional complex projective space. By construction, E has a natural Hermitian metric h_1 . On the other hand, stability implies the existence of a Hermitian-Einstein metric in E which is unique up to a positive scalar. Now the obvious question is if h_1 is in fact the Hermitian-Einstein metric. In this note we indicate how to show by computation that this is not the case.

1 Introduction and main result

Let N be a null correlation bundle on complex projective 3-space \mathbb{P}_3 , i.e. a quotient of $\Omega_{\mathbb{P}_3}^1$ by $\mathcal{O}_{\mathbb{P}_3}(-1)$ (see e.g. [OSS]). Then N is stable in the sense of [OSS], or equivalently, g -stable in the sense of [LT], where g is the Fubini-Study metric in \mathbb{P}_3 , so the Kobayashi-Hitchin correspondence tells us that there exists a g -Hermitian-Einstein metric h_0 in N , which is unique up to a constant positive factor. On the other hand, the standard metric in \mathbb{C}^4 not only induces the Fubini-Study metric in \mathbb{P}_3 , but also natural metrics in $\Omega_{\mathbb{P}_3}^1$ and $\mathcal{O}_{\mathbb{P}_3}(-1)$, and hence a metric h_1 in the quotient N , too. Now the obvious question arises:

(Q) Does it hold $h_1 = c \cdot h_0$ with a positive constant c , or equivalently, does h_1 satisfy the g -Hermitian-Einstein equation

$$(HE) \quad K_{h_1} = \lambda \cdot \text{id}_E$$

where K_{h_1} is the mean curvature of h_1 and λ a real constant?

This question was answered in the affirmative in [L] by manual computations with respect to local coordinates and a local holomorphic frame field.

In this note we consider the following similar situation. It is well known that on the 4-dimensional complex projective space $\mathbb{P}_4 = \mathbb{P}(\mathbb{C}^5)$ there exists a stable holomorphic 2-bundle E with Chern numbers $c_1(E) = 5$ and $c_2(E) = 10$, the *Horrocks-Mumford bundle* [HM], [OSS]. Again, stability of E in the sense of [OSS] is the same as g -stability in the sense of [LT], where g is the Fubini-Study metric in \mathbb{P}_4 , so there exists a g -Hermitian-Einstein metric h_0 in E , which is unique up to a constant positive factor. On the other hand, using the construction of E given in [OSS] one gets in a natural way an explicit metric h_1 in E induced by the standard metric in \mathbb{C}^5 , hence question (Q) arises for E , too. Again we used explicit calculations in local coordinates to tackle this problem, and the result is

Theorem 1 *The metric natural metric h_1 in the Horrocks-Mumford bundle is **NOT** g -Hermitian-Einstein.*

In section 2 we sketch our approach to the problem, and in section 3 we give some details and explicit formulae which should be sufficient to make our calculations reproducible.

2 Our approach

The construction in [OSS] we use does not produce the bundle E directly, but the bundle $E(-2) = E \otimes \mathcal{O}_{\mathbb{P}_4}(-2)$ and a metric h in it. Let h_2 be the standard metric in $\mathcal{O}_{\mathbb{P}_4}(2) = \mathcal{O}_{\mathbb{P}_4}(1)^{\otimes 2}$, induced by the canonical inclusion $\mathcal{O}_{\mathbb{P}_4}(1)^* = \mathcal{O}_{\mathbb{P}_4}(-1) \hookrightarrow \mathbb{P}_4 \times \mathbb{C}^5$ and the standard metric in \mathbb{C}^5 , then the natural metric h_1 in $E = E(-2) \otimes \mathcal{O}_{\mathbb{P}_4}(2)$ is the metric induced by h and h_2 .

Our initial guess was that h_1 would be indeed g -Hermitian-Einstein. Since h_2 is known to be g -Hermitian-Einstein, this is equivalent to h being g -Hermitian-Einstein. Hence we attempted to show that the equation (HE) holds for h ; by continuity, it suffices to do that on some open dense subset U_0^* of \mathbb{P}_4 . So in a suitable chart for \mathbb{P}_4 we explicitly determined a matrix representation H of the metric h with respect to a holomorphic frame field; this already involved algebraic calculations which where impossible to do by hand (H contains rational expressions in the 8 real variables $x_1, \bar{x}_1, \dots, x_4, \bar{x}_4$, with numerators of degree up to 16), so we used the computer package MAPLE. The next step would have been the calculation of the mean curvature, i.e. essentially the matrix $K = (K_{ij})_{i,j=1}^2$ where

$$K_{ij} = - \sum_{\alpha, \beta=1}^4 g^{\beta\alpha} \left(\sum_{k=1}^2 \frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta} H^{kj} - \sum_{k,l,m=1}^2 \frac{\partial H_{ik}}{\partial x_\alpha} H^{kl} \frac{\partial H_{lm}}{\partial \bar{x}_\beta} H^{mj} \right).$$

Here the $g_{\alpha\beta}$ are the coefficients of the Fubini-Study metric with respect to the local holomorphic coordinates x_α , and upper indices mean coefficients of the inverse matrix. Now if the metric h was g -Hermitian, then since $c_1(E(-2)) = 1$ this would be equivalent to $K(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ for all $x \in U_0^*$. (Notice that the constant λ in (HE) is determined by the topology of E (see e.g. [LT]) and can therefore be determined a priori.)

Unfortunately, MAPLE was not able (at least on our computer) to calculate K in a general point x (the main problem being the inverse of H), so we decided to do some testing.

For this, we first let MAPLE determine the derivatives involved in the formula for K_{ij} in a general point. Then we took the particular point $x_0 = (x_1^0, \dots, x_4^0) = (1, 1, 1, 1)$, and could calculate $K(x^0)$ (inversion of the scalar matrix $H(x^0)$ is easy). The result was (as we had hoped) indeed

$$K(x^0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

We repeated the procedure with the point $x^1 = (2, 1, 1, 1)$, and again we got

$$K(x^1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

This seemed to indicate that we had in fact some chance to be right with our first guess.

In the meantime, we had started a different test by looking at the determinant line bundle $L := \det E(-2)$. The induced metric $\det h$ in L is given over U_0^* by the function $\det H$, and if h was g -Hermitian-Einstein, then $\det h$ would be g -Hermitian-Einstein, too; more precisely, the mean curvature $K_{\det H}$ of $\det h$ would be the constant function 4. Again we got a problem: MAPLE could calculate $\det H$, but was not able to simplify the resulting rational function to a form from which it could determine $K_{\det H}$. But we made the motivated guess that

$$\det H = \frac{|x_1|^2|x_2|^2|x_3|^2|x_4|^2}{1 + \|x\|^2},$$

and were able (using MAPLE) to verify the correctness of this formula. Now it was easy to check (even by hand) that indeed $K_{\det H} \equiv 4$, i.e. that the induced metric in $\det E(-2)$, and hence that in $\det E$, is g -Hermitian-Einstein.

So far everything seemed to be okay, but testing of a third point $x^2 = (1 + i, 1, 1, 1)$ gave the disappointing result

$$K(x^2) = \begin{pmatrix} 2 & -\frac{217}{25992} \\ -\frac{217}{25992} & 2 \end{pmatrix}!$$

This of course meant precisely what we did not want to show, namely that the metric in $E(-2)$, and hence the metric in E , is not g -Hermitian-Einstein.

3 Some details and formulae

The bundle $E(-2)$ is the cohomology of a monad

$$(M) \quad 0 \rightarrow \mathbb{C}^5 \otimes \mathcal{O}_{\mathbb{P}_4} \xrightarrow{a} (\Lambda^2 Q)^{\oplus 2} \xrightarrow{b} (\mathbb{C}^5)^* \otimes \Lambda^4 Q \rightarrow 0 ,$$

i.e. a is injective, b is surjective, it holds $\text{im}(a) \subset \ker(b)$, and $E(-2) = \ker(b)/_{\text{im}(a)}$; here $Q = T_{\mathbb{P}_4}(-1)$ (see [OSS]). Let $\pi : \mathbb{C}^5 \otimes \mathcal{O}_{\mathbb{P}_4} \rightarrow Q$ be the natural projection in the Euler sequence. The standard Hermitian inner product in \mathbb{C}^5 defines the standard flat Hermitian metric in the trivial bundle $\mathbb{C}^5 \otimes \mathcal{O}_{\mathbb{P}_4}$, and hence a quotient metric h_Q in Q . This induces a metric $\Lambda^2 h_Q$ in $\Lambda^2 Q$, and hence a metric h_3 in $(\Lambda^2 Q)^{\oplus 2}$ by taking the two summands as orthogonal. Next we get a metric h_b in $\ker(b)$ by restricting h_3 , and finally a quotient metric h in $E(-2)$.

Let $x = (x_0 : x_1 : \dots : x_4)$ be the homogeneous coordinates in \mathbb{P}_4 with respect to the standard basis e_0, \dots, e_4 of \mathbb{C}^5 . The holomorphic section in $\mathbb{C}^5 \otimes \mathcal{O}_{\mathbb{P}_4}$ defined by e_i is denoted \tilde{e}_i , and we define $v_i := \pi(\tilde{e}_i) \in H^0(\mathbb{P}_4, Q)$, $i = 0, \dots, 4$.

Over $U_0 := \{x \in \mathbb{P}_4 \mid x_0 \neq 0\}$, $\underline{v} := (v_1, \dots, v_4)$ is a holomorphic frame field for Q . For $x = (1 : x_1 : \dots : x_4) \in U_0$ the quotient metric in $Q(x)$ is given by

$$h_Q(v_i, v_j)(x) = \delta_{ij} - \frac{\bar{x}_i x_j}{n} , \quad 1 \leq i, j \leq 4 ,$$

where $n = 1 + \sum_{i=1}^4 |x_i|^2$. A holomorphic frame field $\underline{u} = (u_1, \dots, u_6)$ for $\Lambda^2 Q$ over U_0 is given by $u_1 := v_1 \wedge v_2$, $u_2 := v_1 \wedge v_3$, $u_3 := v_1 \wedge v_4$, $u_4 := v_2 \wedge v_3$, $u_5 := v_2 \wedge v_4$, $u_6 := v_3 \wedge v_4$. Since $\Lambda^2 h_Q(v_i \wedge v_j, v_k \wedge v_l) = h_Q(v_i, v_k)h_Q(v_j, v_l) - h_Q(v_i, v_l)h_Q(v_j, v_k)$, it is easy to determine the matrix representation of $\Lambda^2 h_Q(x)$ with respect to $u(x)$. The holomorphic frame field $\underline{b} := (b_1, \dots, b_{12})$ for $(\Lambda^2 Q)^{\oplus 2}$ is defined by $b_i := (u_i, 0)$ for $1 \leq i \leq 6$ and $b_i := (0, u_{i-6})$ for $7 \leq i \leq 12$; then the matrix representation of $h_3(x)$ with respect to $\underline{b}(x)$ is $h_3(x) = \begin{pmatrix} \Lambda^2 h_Q(x) & 0 \\ 0 & \Lambda^2 h_Q(x) \end{pmatrix}$.

Define $a_{\pm} : \mathbb{C}^5 \rightarrow \Lambda^2 \mathbb{C}^5$ by $a_+(e_i) := e_{i+2} \wedge e_{i+3}$, $a_-(e_i) := e_{i+1} \wedge e_{i+4}$, $0 \leq i \leq 4$ (indices *mod* 5). Then the map a in (M) is defined as the composition

$$a(x) : \mathbb{C}^5 \xrightarrow{(a_+, a_-)} (\Lambda^2 \mathbb{C}^5)^{\oplus 2} \xrightarrow{(\Lambda^2 \pi(x))^{\oplus 2}} (\Lambda^2 Q(x))^{\oplus 2} .$$

Using $\pi(x)(e_0) = -\sum_{i=1}^4 x_i v_i$, it follows that the basis (a_3, \dots, a_7) , $a_i := a(e_i)$, of $\text{im}(a)$ is given in coordinates with respect to \underline{b} as

$$\begin{aligned} a_3(x) &= (0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0) , \\ a_4(x) &= (0, 0, 0, 0, 0, 1, x_1, 0, 0, -x_3, -x_4, 0) , \\ a_5(x) &= (0, 0, x_1, 0, x_2, x_3, 0, -1, 0, 0, 0, 0) , \\ a_6(x) &= (x_2, x_3, x_4, 0, 0, 0, 0, 0, 0, 0, -1, 0) , \\ a_7(x) &= (1, 0, 0, 0, 0, 0, 0, 0, -x_1, 0, -x_2, -x_3) . \end{aligned}$$

The map $b : (\Lambda^2 Q)^{\oplus 2} \longrightarrow (\mathbb{C}^5)^* \otimes \Lambda^4 Q = \text{Hom}(\mathbb{C}^5 \otimes \mathcal{O}_{\mathbb{P}^5}, \Lambda^4 Q)$ in (M) is defined by

$$b(x)(\xi, \eta)(v) := -\eta \wedge (\Lambda^2 \pi(x))(a_+(v)) + \xi \wedge (\Lambda^2 \pi(x))(a_-(v))$$

for $v \in \mathbb{C}^5$ and $\xi, \eta \in \Lambda^2 Q(x)$. It is easily checked that the vectors

$$\begin{aligned} a_1(x) &= (x_1 x_2, 0, x_1 x_4, 0, 0, x_3 x_4, 0, 0, 0, 0, 0, 0) , \\ a_2(x) &= (0, 0, 0, 0, 0, 0, 0, x_1 x_3, 0, x_2 x_3, x_2 x_4, 0) \end{aligned}$$

are in $\ker(b(x))$, and that $\underline{a} := (a_1, \dots, a_7)$ is a holomorphic frame field for $\ker(b)$ over $U_0^* := \{x \in U_0 \mid x_i \neq 0, 1 \leq i \leq 4\}$. Since (a_3, \dots, a_7) is a basis of $\text{im}(a)$, the projection $\ker(b) \longrightarrow \ker(b)/_{\text{im}(a)} = E(-2)$ maps a_1, a_2 to a holomorphic frame field $\tilde{\underline{a}} := (\tilde{a}_1, \tilde{a}_2)$ of $E(-2)$ over U_0^* . We write the matrix representation of $h_3(x)|_{\ker(b)}$ with respect to \underline{a} as block matrix

$$h_3(x)|_{\ker(b)} = \frac{1}{n} \begin{pmatrix} C & \bar{B}^t \\ B & A \end{pmatrix} .$$

where A is the 5×5 -matrix representing $h_3(x)|_{\text{im}(a)}$. This can be calculated explicitly, using the matrix for $h_3(x)$; the result is

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

where

$$\begin{aligned} c_1 &= |x_1|^2 |x_2|^2 (1 + |x_3|^2) + |x_1|^2 |x_4|^2 + |x_3|^2 |x_4|^2 (1 + |x_2|^2) , \\ c_2 &= |x_1|^2 |x_3|^2 (1 + |x_4|^2) + |x_2|^2 |x_3|^2 + |x_2|^2 |x_4|^2 (1 + |x_1|^2) , \end{aligned}$$

$$B = \begin{pmatrix} (|x_1|^2 + |x_4|^2) \bar{x}_2 \bar{x}_3 & -(|x_2|^2 + |x_3|^2) \bar{x}_1 \bar{x}_4 \\ (1 + |x_2|^2) \bar{x}_3 \bar{x}_4 & -(|x_3|^2 + |x_4|^2) \bar{x}_2 \\ (|x_1|^2 + |x_3|^2) \bar{x}_4 & -(1 + |x_4|^2) \bar{x}_1 \bar{x}_3 \\ (|x_2|^2 + |x_4|^2) \bar{x}_1 & -(1 + |x_1|^2) \bar{x}_2 \bar{x}_4 \\ (1 + |x_3|^2) \bar{x}_1 \bar{x}_2 & -(|x_1|^2 + |x_2|^2) \bar{x}_3 \end{pmatrix} ,$$

and

$$A = \begin{pmatrix} n+1 & \bar{x}_2 x_4 & \bar{x}_4 x_3 & \bar{x}_1 x_2 & \bar{x}_3 x_1 \\ \bar{x}_4 x_2 & n+|x_1|^2 & \bar{x}_3 & x_4 & \bar{x}_2 x_3 \\ \bar{x}_3 x_4 & x_3 & n+|x_2|^2 & \bar{x}_4 x_1 & \bar{x}_1 \\ \bar{x}_2 x_1 & \bar{x}_4 & \bar{x}_1 x_4 & n+|x_3|^2 & x_2 \\ \bar{x}_1 x_3 & \bar{x}_3 x_2 & x_1 & \bar{x}_2 & n+|x_4|^2 \end{pmatrix}.$$

The matrix representation of the metric $h(x)$ in $E(x)$ with respect to $\underline{\tilde{a}}(x)$ is now given by

$$h(x) = \frac{1}{n}(C - \bar{B}^t \cdot A^{-1} \cdot B).$$

We used MAPLE to explicitly calculate $h(x)$, but the resulting expression is too large to write down here.

We view $x = (x_1, \dots, x_4)$ as holomorphic coordinates in U_0 via the standard chart $(x_1, \dots, x_4) \mapsto (1 : x_1 : \dots : x_4)$. With respect to these coordinates, the Kähler form of the Fubini-Study metric g is $\omega_g = \frac{i}{2} \sum_{\alpha, \beta=1}^4 g_{\alpha\beta} dx_\alpha \wedge d\bar{x}_\beta$, where $g_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{n} - \frac{\bar{x}_\alpha x_\beta}{n^2}$ with

$$n = 1 + \sum_{i=1}^4 |x_i|^2 \text{ as above.}$$

Let D be the Chern connection in $(E(-2), h)$, i.e. the unique h -unitary connection compatible with the holomorphic structure in $E(-2)$ (compare [K], [LT]), and $F = D \circ D$ its curvature. With respect to the holomorphic frame field $\underline{\tilde{a}}$ for $E(-2)$ over U_0^* , we write $F = (F_{ij})_{i,j=1,2}$ and $F_{ij} = \sum_{\alpha, \beta=1}^4 F_{ij\alpha\beta} dx_\alpha \wedge d\bar{x}_\beta$, $1 \leq i, j \leq 2$. Let be $h = (H_{ij})_{i,j=1,2}$ with respect to $\underline{\tilde{a}}$, and $(H^{ij})_{i,j=1,2} := h^{-1}$. Then it holds

$$F_{ij\alpha\beta} = - \sum_{k=1}^2 \frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta} H^{kj} + \sum_{k,l,m=1}^2 \frac{\partial H_{ik}}{\partial x_\alpha} H^{kl} \frac{\partial H_{lm}}{\partial \bar{x}_\beta} H^{mj}.$$

The mean curvature K of h (with respect to g) is defined by the relation

$$F \wedge \omega_g^3 = -\frac{i}{2} K \omega_g^4.$$

With respect to $\underline{\tilde{a}}$ we write $K = (K_{ij})_{i,j=1,2}$, then it holds

$$(*) \quad K_{ij} = \sum_{\alpha, \beta=1}^4 g^{\beta\alpha} F_{ij\alpha\beta} = - \sum_{\alpha, \beta=1}^4 g^{\beta\alpha} \left(\sum_{k=1}^2 \frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta} H^{kj} - \sum_{k,l,m=1}^2 \frac{\partial H_{ik}}{\partial x_\alpha} H^{kl} \frac{\partial H_{lm}}{\partial \bar{x}_\beta} H^{mj} \right).$$

where $(g^{\alpha\beta})_{\alpha, \beta=1, \dots, 4} := ((g_{\alpha\beta})_{\alpha, \beta=1, \dots, 4})^{-1}$, i.e. $g^{\alpha\beta} = n(\delta_{\alpha\beta} + \bar{x}_\alpha x_\beta)$.

Since the H_{ij} and $g^{\alpha\beta}$ are explicitly given, the calculation of $K(x^0)$ for a given point x^0 can now be done as follows (using MAPLE where necessary):

- determine $\frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta}(x)$, $\frac{\partial H_{ik}}{\partial x_\alpha}(x)$, $\frac{\partial H_{ik}}{\partial \bar{x}_\beta}(x)$, $1 \leq i, k \leq 2$, $1 \leq \alpha, \beta \leq 4$, for a general point x ;
- substitute x^0 into h , and invert the scalar matrix $h(x^0)$ to get the $H^{ij}(x^0)$'s;
- substitute x^0 into $\frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta}$, $\frac{\partial H_{ik}}{\partial x_\alpha}$, $\frac{\partial H_{ik}}{\partial \bar{x}_\beta}$, $g^{\alpha\beta}$, $1 \leq i, k \leq 2$, $1 \leq \alpha, \beta \leq 4$;
- substitute the resulting scalars into the right hand side of equation (*), and evaluate.

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